MATH 2050C Lecture 7 (Feb 2)

Established GOAL: $\mathbb{R}$ is the complete ordered field. (Ch.2)

Sequences and their limits (Ch. 3)

GOAL: Define "$\lim_{n\to\infty}(x_n) = x$" & study limit properties

Q: What is a "sequence" of real numbers?

Def$^{\underline{n}}$: A sequence of real numbers is a function

$$X : \mathbb{N} \longrightarrow \mathbb{R}$$

Denote: $X(1) =: x_1, \quad X(2) =: x_2, \ldots, \quad X(n) =: x_n$

Write: $X = (x_n) = (x_1, x_2, x_3, \ldots)$

BE CAREFUL: Sets VS Sequences

E.g.) $\left((-1)^n\right) = (-1, 1, -1, 1, -1, 1, \ldots\ldots)$ ordered & infinite

$\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ unordered & finite

Examples: (1) constant seq. $(1, 1, 1, 1, 1, \ldots)$

(2) geometric seq. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right) = \left(\frac{1}{2^n}\right)$

(3) arithmetic seq. $(1, 4, 7, 10, 13, \ldots) = (3n-2)$

$$\begin{cases} \text{odd seq.} & (1, 3, 5, 7, \ldots) \\ \text{even seq.} & (0, 2, 4, 6, 8, \ldots) \end{cases}$$

(4)* Fibonacci seq. (inductively defined)

$$x_1 := 1 \quad ; \quad x_2 := 1$$

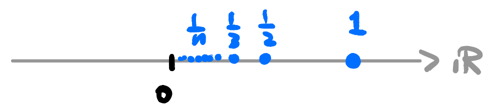
$$x_n := x_{n-1} + x_{n-2} \qquad \text{for } n \geq 3$$

$$(x_n) = (1, 1, 2, 3, 5, 8, 13, 21, \dots \dots)$$

Motivational Example

$$\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{10000}, \dots\right)$$

Idea: $\frac{1}{n}$ "$\rightarrow$" 0 as n "$\rightarrow$" ∞



Q: Can we describe this mathematically?

Def$^{\underline{n}}$: (ε - K def$^{\underline{n}}$ for limit)

We say that (x_n) converges to $x \in \mathbb{R}$

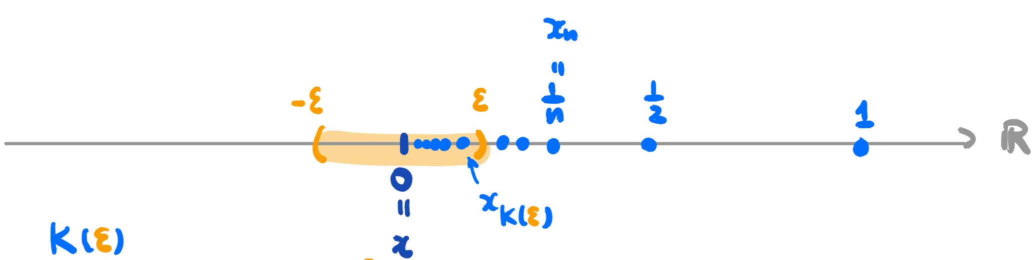
Written: $\lim (x_n) = x$ or $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$

iff $\forall \underset{\text{(small)}}{\varepsilon} > 0$, $\exists \underset{\text{(large)}}{K(\varepsilon)} \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \qquad \forall\, n \geq K(\varepsilon).$$

(i.e. $x - \varepsilon < x_n < x + \varepsilon$)

Picture:



Idea: x_n $\overbrace{\text{"eventually"}}^{K(\varepsilon)}$ are $\overbrace{\text{"close"}}^{\varepsilon}$ to x

Example: $\boxed{\lim\left(\frac{1}{n}\right) = 0}$

Proof: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K(\varepsilon) \in \mathbb{N}$ s.t. $K(\varepsilon) > \frac{1}{\varepsilon}$. (by Archimedean Property)

Then, $\forall\, n \geq K(\varepsilon)$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{K(\varepsilon)} < \varepsilon$$

$\blacksquare$

Example: $\boxed{\lim\left(\frac{3n+2}{n+1}\right) = 3}$

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $\underbrace{K > \frac{1}{\varepsilon}}_{(*)}$ (by Archimedean Property).

Then, $\forall\, n \geq K$,

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{(3n+2) - 3(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right|$$

$$= \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{K} \overset{(*)}{<} \varepsilon$$

$\blacksquare$

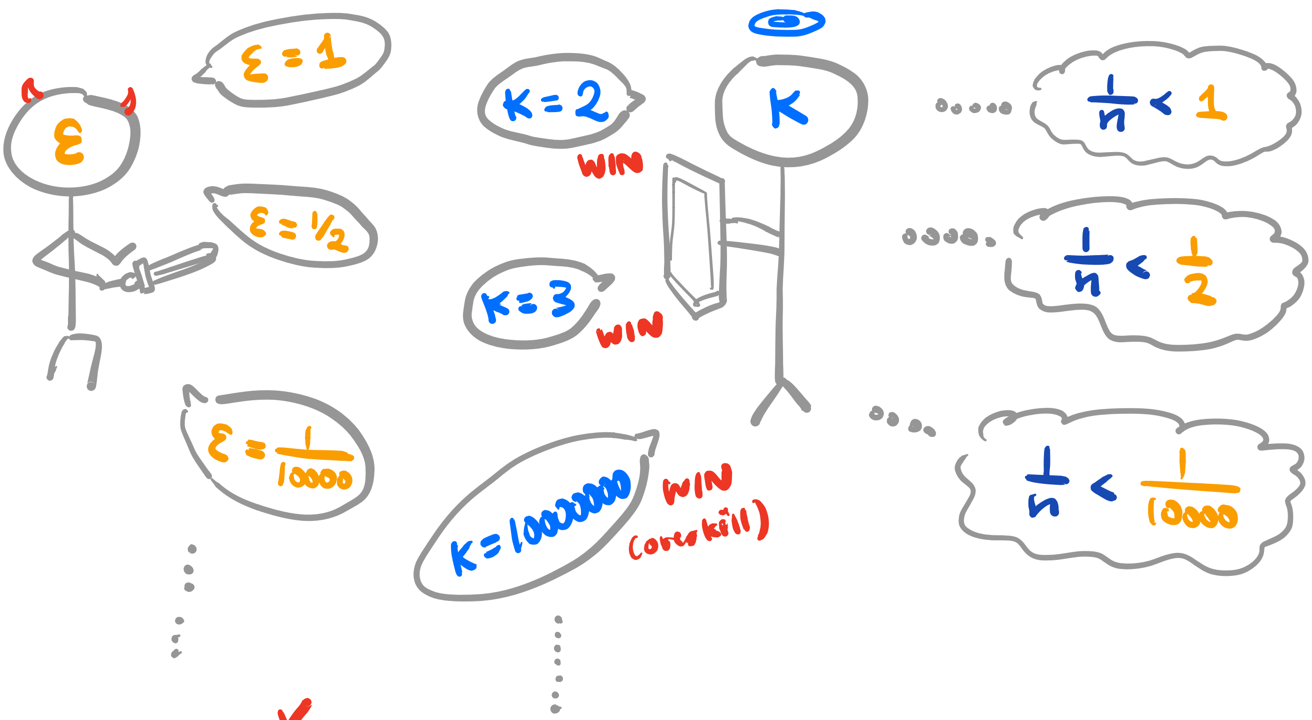
Def$^{\underline{n}}$: Given a seq (x_n) of real numbers, we say

(i) (x_n) is **convergent** if $\exists\, x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

(ii) (x_n) is **divergent** if (x_n) is <u>NOT</u> convergent

i.e. $\nexists\, x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

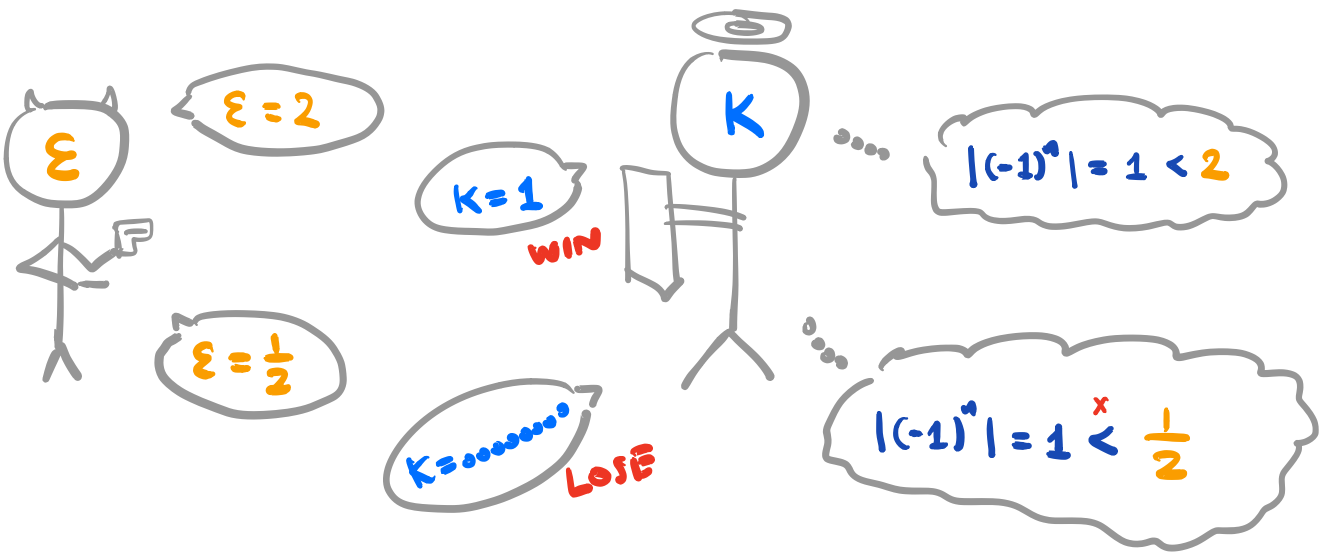
Let's consider the example $\lim\left(\frac{1}{n}\right) = 0$ again :



$\lim\left(\frac{1}{n}\right) = 0$ ✓ convergent!

As another example, consider $(x_n) = \left((-1)^n\right)$

Q: Is $\lim(x_n) = 0$?



$\Rightarrow \quad \lim(x_n) \neq 0$

Q: Is $\lim(x_n) = 1$?

$$\Rightarrow \quad \lim(x_n) \neq 1$$

FACT: For this seq., $\nexists\ x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

Prop: Any convergent seq. (x_n) has a unique limit.

Proof: Suppose NOT. Then, $\exists\ x \overset{(*)}{\neq} x' \in \mathbb{R}$ s.t.

$$\lim(x_n) = x \quad \text{AND} \quad \lim(x_n) = x'$$

Consider $\varepsilon \overset{(**)}{:=} \dfrac{|x - x'|}{4} \overset{(*)}{>} 0$. According to ε-K def^n.

$\lim(x_n) = x \Rightarrow \exists\ K(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \qquad \forall n \geq K$$

$\lim(x_n) = x' \Rightarrow \exists\ K'(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x'| < \varepsilon \qquad \forall n \geq K'$$

Take $\overline{K} := \max\{K, K'\} \in \mathbb{N}$.

By Triangle ineq.$^{(\#)}$ $\forall n \geq K$.

$$|x - x'| = |(x_n - x) - (x_n - x')|$$
$$\overset{(\#)}{\leq} |x_n - x| + |x_n - x'|$$
$$< \varepsilon + \varepsilon \overset{(*)}{=} \frac{1}{2}|x - x'|$$

Contradiction!

More Examples

(a) Let $a > 0$. Then $\boxed{\lim\left(\frac{1}{1+na}\right) = 0}$.

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K \overset{(*)}{>} \frac{1}{a\varepsilon}$

Then, $\forall n \geq K$,

$$\left|\frac{1}{1+na} - 0\right| = \frac{1}{1+na} < \frac{1}{na}$$
$$\leq \frac{1}{Ka} \overset{(*)}{<} \varepsilon$$

$\square$

$$\left|\frac{1}{1+na} - 0\right| = \frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$
$$K > \frac{1}{a\varepsilon}$$

(b) Let $b \in (0,1)$. Then $\boxed{\lim(b^n) = 0}$.

Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon}$

where $b = \frac{1}{1+a}$ for some $a > 0$.

(A)

$$|b^n - 0| = b^n < \varepsilon$$
$$\Updownarrow$$
$$n \log b < \log \varepsilon$$
$$\Downarrow$$
$$n > \log\varepsilon / \log b$$

Then, $\forall n \geq K$.

Bernoulli : $(1+x)^n \geq 1+nx$
$\forall x \geq -1$

$$|b^n - 0| = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \varepsilon$$

Same as previous eg.

$\blacksquare$

Write $b = \frac{1}{1+a}$, where $a > 0$

$$|b^n - 0| = b^n = \frac{1}{(1+a)^n}$$

$$\leq \frac{1}{1+na} < \varepsilon$$

Bernoulli